Homeomorphic Solutions to Reduced Beltrami Equations

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Abstract

We study differential expressions related to linear families of quasiconformal mappings and give a simple and direct proof to a result due to Alessandrini and Nesi [1].

2000 AMS Mathematics Classification Numbers. Primary 30C62. Secondary 35J45.

Keywords. Quasiconformal mappings, Beltrami operators, Elliptic PDEs, G-closure problems.

1 Introduction

The reduced Beltrami differential equation

$$\frac{\partial f}{\partial \bar{z}} = \lambda(z)\, \Im m \left(\frac{\partial f}{\partial z}\right), \qquad |\lambda(z)| \leqslant k < 1, \tag{1}$$

arises naturally in many different contexts in the theory of quasiconformal mappings; for instance in the Stoilow factorization and the G-closure problems for the general Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z} + \nu(z)\,\overline{\frac{\partial f}{\partial z}}, \qquad |\mu(z)| + |\nu(z)| \leqslant k < 1. \tag{2}$$

The solutions to (1) have a number of special properties, such as $f(z) \equiv z$ being always a solution. This is the unique normalized solution fixing 0 and 1. In fact, if $g \in W^{1,2}_{loc}(\mathbb{C})$ is a homeomorphism satisfying the equation (1) and g has two fixed points, then $g(z) \equiv z$. We refer the reader for these and other properties of the reduced equation to the monograph [2]. An early application of the reduced equation for uniqueness properties of (2) for self-mappings of the unit disk can be found in [3].

Studies of the reduced Beltrami equation (1) indicate that for its solutions, the null Lagrangian

$$\mathcal{J}(z, f) = \Im m \left(\frac{\partial f}{\partial z}\right)$$

has many properties analogous to the familiar Jacobian determinant of a Sobolev mapping. This suggest the following conjecture, cf. [2, p. 222].

*Authors were supported by the Academy of Finland, project no. 1118634, the Finnish Centre of Excellence in Analysis and Dynamics Research, and project MRTN-CT-2006-035651, Acronym CODY, of the European Commission.

Conjecture 1.1. *Suppose $f : \Omega \to \mathbb{C}$ is a (quasiregular) solution to the reduced Beltrami equation* (1)*. Then either $\Im m\big(f_z\big)$ is a constant, or else*

$$\Im m\left(\frac{\partial f}{\partial z}\right) \neq 0 \qquad \textit{almost everywhere in } \Omega.$$

Through the Stoilow factorization type theorems, and [2, Theorem 6.1.1] in particular, the conjecture can equivalently be formulated in terms of solutions to the general Beltrami system (2), see Section 2.

In fact, for homeomorphic solutions the conjecture is closely related to the notion [2, 5] of linear families of quasiconformal mappings. Given a domain $\Omega \subset \mathbb{C}$ and an $\mathbb{R}$-linear subspace $\mathcal{F} \subset W^{1,2}_{loc}(\Omega)$, we say that $\mathcal{F}$ is a *linear family of quasiconformal mappings*, if there is $1 \leqslant K < \infty$ such that for every $g \in \mathcal{F}$, either $g \equiv 0$ or else g is a K-quasiconformal mapping in Ω. It quickly follows [5] that $\dim \mathcal{F} \leqslant 2$. If we have the equality, then

$$\mathcal{F} = \{a\,\Phi + b\,\Psi : \ a, b \in \mathbb{R}\}$$

for some quasiconformal mappings $\Phi : \Omega \to \mathbb{C}$ and $\Psi : \Omega \to \mathbb{C}$. In this case we say that the family $\mathcal{F}$ is *generated* by the mappings Φ and Ψ. In particular, if Φ and Ψ generate a linear family of quasiconformal mappings, then by definition each $F_{a,b} = a\,\Phi + b\,\Psi$ is injective in Ω, whenever $a^2 + b^2 \neq 0$.

In general, quasiconformality is not preserved under linear combinations. However, if we have mappings that happen to be solutions to the same Beltrami equation (2), then their linear combinations are at least quasiregular. Conversely, [5] associates to a linear (two-dimensional) family $\mathcal{F}$ of quasiconformal mappings a Beltrami equation of the type (2) satisfied by every $g \in \mathcal{F}$. The next theorem, answering in positive [5, Conjecture 1], implies that the associated equation is unique.

Theorem 1.2. *Let $\mathcal{F}$ be a linear family of quasiconformal mappings in a domain $\Omega \subset \mathbb{C}$. If $\Phi, \Psi \in \mathcal{F}$, then either*

$$\mathcal{J}(\Phi, \Psi) := \ \Im m\left(\frac{\partial \Phi}{\partial z}\overline{\frac{\partial \Psi}{\partial z}}\right) \neq 0 \qquad \textit{almost everywhere in } \Omega$$

or else

$$a\,\Phi(z) + b\,\Psi(z) \equiv 0 \qquad \textit{for some constants } a, b \in \mathbb{R},\ a^2 + b^2 \neq 0,$$

in which case $\mathcal{J}(\Phi, \Psi) \equiv 0$.

It is possible, in fact, to obtain this theorem by combining results and methods from [1] and [5]. However, the purpose of this paper is to give a simple and direct proof to this beautiful result. Our methods in proving the theorem are similar to those of Alessandrini and Nesi in [1], but we will simplify their approach. For principal solutions the result was also announced in [4], but unfortunately [4, Proposition 2] is not valid, with counterexamples easy to find.

Not every pair of homeomorphic solutions generate a linear family of quasiconformal mappings; simple examples can be found already among the solutions to the Cauchy-Riemann system $f_{\bar{z}} = 0$. For instance, $\Phi(z) = z$ and $\Psi(z) = z^2$

are both conformal in $\Omega = \{z : \Re e(z) > 0\}$, yet (some of) their linear combinations are non-injective in Ω, and thus the mappings do not generate a linear family of (quasi)conformal mappings.

However, global homeomorphic solutions to (2) are determined by their values at two distinct points, and it follows from this fact that in the domain $\Omega = \mathbb{C}$, linear combinations of homeomorphic solutions are either constants or homeomorphisms, see [2, Section 6.2]. Hence Theorem 1.2 applies.

Corollary 1.3. *Suppose $\Phi, \Psi \in W^{1,2}_{loc}(\mathbb{C})$ are homeomorphic solutions to* (2). *Then, unless the mappings are affine combinations of each other,*

$$\Im m \left(\frac{\partial \Phi}{\partial z} \overline{\frac{\partial \Psi}{\partial z}} \right) \neq 0 \qquad \textit{almost everywhere in } \mathbb{C}.$$

As an immediate consequence, Conjecture 1.1 follows for global homeomorphic solutions $f : \mathbb{C} \to \mathbb{C}$ to the reduced equation (1).

There are further situations where the injectivity of a linear family of solutions to (2) can be guaranteed. For example, if f satisfies (2) in a bounded convex domain Ω with

$$\Re e\big(f(z)\big) = \Re e\big(\mathcal{A}(z)\big) \quad \text{on } \partial\Omega, \qquad \mathcal{A} : \mathbb{C} \to \mathbb{C} \ \text{ a linear isomorphism,}$$

then f is injective: With the Stoilow factorization we can write $f = h \circ F^{-1}$, where h is holomorphic in the unit disk $\mathbb{D}$ and $F : \mathbb{D} \to \Omega$ is a quasiconformal homeomorphism. Since $\mathcal{A} \circ F$ maps to a convex domain, by the classical Radó-Kneser-Choquet theorem the Poisson extension U of its boundary values is one-to-one. But $\Re e\big(h\big) = \Re e\big(U\big)$ on $\partial\mathbb{D}$, hence in $\mathbb{D}$, and by the theorem of Clunie and Sheil-Small [6, Theorem 5.3], or [7, p. 38], the injectivity of h and f follows. For alternative proofs of injectivity, using properties of the Beltrami equation, see [1], [5] or [10]. We now obtain the following result of Alessandrini and Nesi in [1].

Corollary 1.4. *Suppose $\Omega \subset \mathbb{C}$ is a bounded convex domain, and let $\Phi, \Psi \in W^{1,2}(\Omega)$ be solutions to* (2) *in Ω, such that*

$$\Re e\big(\Phi(z) - z\big) \in W^{1,2}_0(\Omega), \qquad \Re e\big(\Psi(z) + iz\big) \in W^{1,2}_0(\Omega).$$

Then

$$\Im m \left(\frac{\partial \Phi}{\partial z} \overline{\frac{\partial \Psi}{\partial z}} \right) > 0 \qquad \textit{almost everywhere in } \Omega.$$

According to [5, Lemma 7.1] we have $\Im m\big(\Phi_z \overline{\Psi_z}\big) \geqslant 0$ almost everywhere. Hence the corollary follows from Theorem 1.2.

Finally we note that the quantity $\mathcal{J}(\Phi, \Psi)$ arises naturally in the study [5, 9] of the G-closure problems of Beltrami operators, and this connection was also the motivation in the work of Alessandrini and Nesi. In fact, combining Theorem 1.2 with results and ideas developed in [5] and [9], we see that the family $\mathcal{F}_K(\mathbb{C})$, $1 \leqslant K < \infty$, of Beltrami differential operators

$$\frac{\partial}{\partial \bar{z}} - \mu(z)\frac{\partial}{\partial z} - \nu(z)\overline{\frac{\partial}{\partial z}}$$

with

$$|\mu(z)| + |\nu(z)| \leqslant \frac{K-1}{K+1} = k < 1, \qquad z \in \mathbb{C},$$

is G-compact. For the details, we refer the reader to [2, Chapter 16].

2 Proof of Theorem 1.2

We start by reducing Theorem 1.2 to the reduced Beltrami equation, using basic facts from [2] and [5]. For this we may assume that $\Phi, \Psi \in W^{1,2}_{loc}(\Omega)$ generate the linear family $\mathcal{F}$ of K-quasiconformal mappings. According to [5, Section 5.3] or [2, Remark 16.6.7], there are Beltrami coefficients μ and ν such that every element $g \in \mathcal{F}$ satisfies the equation

$$\frac{\partial g}{\partial \bar{z}} = \mu(z)\frac{\partial g}{\partial z} + \nu(z)\,\overline{\frac{\partial g}{\partial z}} \qquad \text{almost everywhere in } \Omega, \tag{3}$$

where

$$|\mu(z)| + |\nu(z)| \leqslant \frac{K-1}{K+1} = k < 1.$$

Next, following [5, Lemma 7.1], we apply the fact that Φ and Ψ generate a linear family of injections. Since for every $a, b \in \mathbb{R}$ the mappings $a\,\Phi(z) + b\,\Psi(z)$ are injections, we have

$$\Lambda(z,w) := \Im m\left(\frac{\Phi(z)-\Phi(w)}{\Psi(z)-\Psi(w)}\right) \neq 0, \qquad z, w \in \Omega,\ z \neq w.$$

As the complement of the diagonal $\{(z,z) : z \in \Omega\}$ is connected in $\Omega \times \Omega$, the continuous function $\Lambda(z,w)$ does not change sign. We may assume that $\Lambda(z,w) < 0$ whenever $z \neq w$; otherwise replace Ψ by $-\Psi$. From this fact and Taylor's first-order development, we obtain at points z of differentiability, thus almost everywhere, that

$$\Im m\left(\Phi_z(z)\,\overline{\Psi_z(z)}\right) \leqslant 0.$$

The explicit details can be found in [5, Lemma 7.1] and in [2, p. 203].

We now arrive at a reduced equation. Namely,

$$\Psi(z) = (f \circ \Phi)(z), \qquad z \in \Omega,$$

for some quasiconformal homeomorphism $f : \Phi(\Omega) \to \Psi(\Omega)$. The general Stoilow factorization theorem [2, Theorem 6.1.1] states that, since Φ and Ψ satisfy the same equation (3), the mapping f is a solution to the reduced equation,

$$\frac{\partial f}{\partial \bar{w}} = \lambda(w)\,\Im m\left(\frac{\partial f}{\partial w}\right), \qquad w \in \Phi(\Omega).$$

Here $\lambda(w) = -2i\nu(z)/(1+|\nu(z)|^2 - |\mu(z)|^2)$ and $w = \Phi(z)$. Furthermore, by the ellipticity bounds in (3), $|\lambda(w)| \leqslant k' = 2k/(1+k^2) < 1$.

With the chain rule one calculates that

$$J(z,\Phi)\,\Im m\big(f_w \circ \Phi\big) = (-1 + |\mu|^2 - |\nu|^2)\,\Im m\big(\Phi_z \overline{\Psi_z}\,\big) \geqslant 0 \tag{4}$$

almost everywhere. In particular, as quasiconformal mappings preserve Lebesgue zero sets, $\Im m\big(f_w\big) \geqslant 0$ almost everywhere in $\Omega' = \Phi(\Omega)$. Moreover, Theorem 1.2 is equivalent to showing that $\Im m\big(f_w\big)$ can vanish only in a set of measure zero.

With this reduction, we are now left to study the homeomorphic solution $f \in W^{1,2}_{loc}(\Omega')$ to the reduced equation (1). Let us write $f(z) = u(z) + iv(z)$, where u and v are real valued. Similarly write $\lambda(z) = \alpha(z) + i\beta(z)$.

Taking the imaginary part of (1) shows us that $u_y + v_x = \beta(v_x - u_y)$, i.e.

$$u_y = \frac{\beta - 1}{\beta + 1}\, v_x.$$

Thus

$$2\,\Im m\left(\frac{\partial f}{\partial z}\right) = v_x - u_y = \frac{2}{\beta + 1}\, v_x = \frac{2}{\beta - 1}\, u_y. \tag{5}$$

Since $|\beta(z)| \leqslant |\lambda(z)| \leqslant k < 1$, the coefficients $2/(\beta(z) \pm 1)$ in (5) are uniformly bounded below. Hence to prove Theorem 1.2 it suffices to show that $u_y \neq 0$ almost everywhere.

The trick of the proof is that, for the reduced equation (1), the derivative u_y is a solution to the adjoint equation determined by a non-divergence type operator. To state this more precisely, consider an operator

$$L = \sum_{i,j=1}^{2} \sigma_{ij}(z)\, \frac{\partial^2}{\partial x_i x_j},$$

where $\sigma_{ij} = \sigma_{ji}$ are measurable and the matrix

$$\sigma(z) = \begin{bmatrix} \sigma_{11}(z) & \sigma_{12}(z) \\ \sigma_{12}(z) & \sigma_{22}(z) \end{bmatrix}$$

is uniformly elliptic,

$$\frac{1}{K}|\xi|^2 \leqslant \langle \sigma(z)\xi, \xi\rangle = \sigma_{11}(z)\xi_1^2 + 2\sigma_{12}(z)\xi_1\xi_2 + \sigma_{22}(z)\xi_2^2 \leqslant K|\xi|^2$$

for all $\xi \in \mathbb{C}$ and $z \in \Omega'$. Here K is the ellipticity constant. Then we say that the function $w \in L^2_{loc}(\Omega')$ is a weak solution to the adjoint equation

$$L^* w = 0 \tag{6}$$

if

$$\int w L\varphi = 0, \qquad \text{for every } \varphi \in C_0^\infty(\Omega').$$

To identify u_y as a solution to an equation of the type (6), we recall that the components of solutions $f = u + iv$ to general Beltrami equations satisfy a divergence type second-order equation, see [2, Section 16.1.5]. In case of (1), it turns out that the component u satisfies the equation

$$\operatorname{div} A\nabla u = 0, \qquad A(z) := \begin{bmatrix} 1 & a_{12}(z) \\ 0 & a_{22}(z) \end{bmatrix}, \tag{7}$$

where the matrix elements are

$$a_{12} = \frac{2\,\Re e(\lambda)}{1-\Im m(\lambda)} = \frac{2\,\alpha}{1-\beta}, \quad a_{22} = \frac{1+\Im m(\lambda)}{1-\Im m(\lambda)} = \frac{1+\beta}{1-\beta} > 0. \tag{8}$$

Precisely, (7) means that for every $\varphi \in C_0^\infty(\Omega')$ we have

$$0 = \int \nabla\varphi \cdot A\nabla u = \int \varphi_x(u_x + a_{12}u_y) + \varphi_y a_{22} u_y. \tag{9}$$

But since derivatives of smooth test functions are again test functions, we can replace φ by $\varphi_y \in C_0^\infty(\Omega')$. In this case the identity (9) takes the form

$$\begin{aligned} 0 &= \int \varphi_{yx}u_x + a_{12}\varphi_{yx}u_y + a_{22}\varphi_{yy}u_y \\ &= \int \varphi_{xx}u_y + a_{12}\varphi_{xy}u_y + a_{22}\varphi_{yy}u_y \\ &= \int (\varphi_{xx} + a_{12}\varphi_{xy} + a_{22}\varphi_{yy})u_y. \end{aligned}$$

Thus u_y is indeed a distributional solution to the adjoint equation $L^*u_y = 0$, where

$$L = \frac{\partial^2}{\partial x^2} + a_{12}\,\frac{\partial^2}{\partial x \partial y} + a_{22}\,\frac{\partial^2}{\partial y^2} \tag{10}$$

and a_{12}, a_{22} are given by (8). Note that the original matrix $A(z)$ is not symmetric. However, the operator L in (10) can be represented by the symmetric matrix

$$\sigma(z) := \begin{bmatrix} 1 & a_{12}(z)/2 \\ a_{12}(z)/2 & a_{22}(z) \end{bmatrix}$$

and as $|\lambda(z)| \leqslant k' < 1$, from (8) we see that σ is uniformly elliptic.

Next, we use (4) and (5) to prove that the derivative $u_y \geqslant 0$ almost everywhere. In fact, it is precisely here we use the assumption that $\mathcal{F}$ consists only of homeomorphisms.

Thus we may assume that u_y is a non-negative solution to the adjoint equation, $L^*u_y = 0$, where L is defined in (10). In this case we may simply apply a result of Fabes and Stroock [8] that the non-negative solutions to the adjoint equation satisfy a uniform reverse Hölder estimate. For the reader's convenience we recall here the explicit formulation of their theorem.

Theorem 2.1. [8, Theorem 2.1] *Consider the operator*

$$L = \sum_{i,j=1}^{2} \sigma_{ij}(z)\,\frac{\partial^2}{\partial x_i x_j},$$

where $\sigma_{ij} = \sigma_{ji}$ are measurable and L is uniformly elliptic with constant K.

Then there exists a constant C_0, depending only on the ellipticity constant K, such that for all $w \geqslant 0$ satisfying $L^*w = 0$ in a domain $\Omega \subset \mathbb{C}$ we have

$$\left[\frac{1}{r^2}\int_{\mathbb{D}(z_0,r)} w(z)^2\,dz\right]^{1/2} \leqslant \frac{C_0}{r^2}\int_{\mathbb{D}(z_0,r)} w(z)\,dz \tag{11}$$

in every disk $\mathbb{D}(z_0,r)$ such that $\mathbb{D}(z_0,2r) \subset \Omega$.

Applying the Fabes-Stroock theorem to $u_y = w$, it follows from (11) that either $w \equiv 0$ or $w > 0$ almost everywhere. Namely, if $E := \{z \in \Omega' : w(z) = 0\}$ has positive measure and $z_0 \in E$ is a point of density, we can find disks $\mathbb{D}_r = \mathbb{D}(z_0, r)$ with

$$|\mathbb{D}_r \setminus E| < \varepsilon r^2, \qquad \text{where } C_0\sqrt{\varepsilon} < 1,$$

and C_0 is the constant of the reverse Hölder inequality (11). Thus

$$\int_{\mathbb{D}_r} w = \int_{\mathbb{D}_r \setminus E} w \leqslant \left(\int_{\mathbb{D}_r} w^2 \right)^{1/2} |\mathbb{D}_r \setminus E|^{1/2} \leqslant C_0 \sqrt{\varepsilon} \int_{\mathbb{D}_r} w. \qquad (12)$$

This is possible only if w vanishes identically in $\mathbb{D}(z_0, r)$, that is $\mathbb{D}(z_0, r) \subset E$. But then we can replace the disk by a slightly larger one, argue as in (12) and by iterating the argument prove that $E = \Omega'$. Now $u_y \equiv 0$, and (1) with (5) shows that f is holomorphic with the real valued derivative, hence affine. Therefore the mappings Φ and $\Psi = f \circ \Phi$ would not generate a linear family of homeomorphisms. Thus $u_y \neq 0$ almost everywhere, and we have completed the proof of Theorem 1.2. □

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